Inelastic collision and switching of coupled bright solitons in optical fibers

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By constructing the general six-parameter bright two-soliton solution of the integrable coupled nonlinear Schrödinger equation (Manakov model) using the Hirota method, we find that the solitons exhibit certain novel inelastic collision properties, which have not been observed in any other $(1+1)$-dimensional soliton system so far. In particular, we identify the exciting possibility of switching solitons between modes by changing the phase. However, the standard elastic collision property of solitons is regained with specific choices of parameters. [S1063-651X(97)13508-5]

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Recent developments in the field of optical solitons in fibers have already revealed the possibility of overcoming the limitations on the speed and distance of linear-wave transmission systems [1,2]. However, the interaction between optical solitons influences directly the capacity and quality of communication [2,3]. Since the nonlinear Schrödinger (NLS) equation is used as a mathematical model for such studies, considerable attention has been devoted to analyze the nature of collision between NLS solitons [1–4]. In more general physical situations [5], coupled NLS equations are often used to describe the interaction among the modes in nonlinear optics, as, for example, in the case of birefringent [6] and other two-mode [7] fibers. Here we consider the integrable coupled NLS equation of Manakov type [8],

$$iq_{1x} + q_{1tt} + 2\mu(|q_1|^2 + |q_2|^2)q_1 = 0,$$

$$iq_{2x} + q_{2tt} + 2\mu(|q_1|^2 + |q_2|^2)q_2 = 0, \qquad (1)$$

where q_1 and q_2 are slowly varying envelopes of the two interacting optical modes, the variables x and t are the normalized distance and time, and μ is a positive parameter.

Exact soliton solutions have been derived [8–11] for the system (1) with different procedures. In [9], using bright one-soliton solutions of the system (1), many physical phenomena such as birefringence property, soliton trapping, and daughter wave (''shadow'') formation are studied. Further, as noted in [12], the Manakov model (1) is important in describing the effects of averaged random birefringence on an orthogonally polarized pulse in a real fiber. In addition, by considering the analytic solution of the system (1), conditions have been established for soliton switching and energy coupling among the two modes in a nonlinear fiber [13]. Recently in [10], two of the present authors (R.R. and M.L.) have derived bright and dark multisoliton solutions of the system (1). The interaction between bright and dark solitary waves is studied in [14].

In this paper, we focus our attention on the system (1) to derive a more general bright two-soliton solution than the ones that have been presented in the literature [10] by using

the Hirota method [15,16]. The asymptotic behavior of this solution is studied further in order to explain the collision properties of coupled solitons in the anomalous dispersion region.

It is often stated in the literature [8,17] that the two-soliton solution is very difficult to derive and too complicated to analyze. (Nevertheless, Manakov [8] was able to make an asymptotic analysis of the solitons.) In this paper we present for the first time the completely general two-soliton solution to the system (1) in explicit form, which can be readily used asymptotic analysis, numerical computations, etc. Our explicit solution not only clarifies Manakov's observation on the asymptotic behavior, but also helps to realize the exciting possibility of a novel type of inelastic collision, allowing switching between components via phase change; this in contrast to the standard elastic collision usually observed in $(1+1)$-dimensional soliton systems. (However, the standard elastic collision property of the solution is recovered when restrictions are imposed on some of the free parameters.) The situation is reminiscent of the dromion solutions in $(2+1)$-dimensional systems such as the Davey-Stewartson equation [18,19] where inelasticity has been observed [19,20] in the scattering process. The details of the present study are as follows.

Recently, a special form of the bright two-soliton solution for Eq. (1) with five arbitrary complex parameters has been derived in [10], using the Hirota method [15,16]. The first step in this method is to transform the system (1) into the Hirota bilinear form. For this purpose we use the transformations [10], $q_1 = g/f$ and $q_2 = h/f$, with f real, to obtain the bilinear form of the system (1) as

$$(iD_x + D_t^2)g \cdot f = 0, \quad (iD_x + D_t^2)h \cdot f = 0,$$

$$D_t^2 f \cdot f = 2\mu(gg^* + hh^*). \qquad (2)$$

Here the Hirota bilinear operator D is defined by $D_x^n a \cdot b = (\partial_{x_1} - \partial_{x_2})a(x_1)b(x_2)|_{x=x_1=x_2}$. The one-soliton solution to Eqs. (2) is given by $f = \alpha e^{\eta_1}$, $h = \beta e^{\eta_1}$, $g = 1 + e^{\eta_1 + \eta_1^* + R}$, that is,

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$$q_1 = \frac{\alpha e^{\eta_1}}{1 + e^{\eta_1 + \eta_1^* + R}}, \quad q_2 = \frac{\beta e^{\eta_1}}{1 + e^{\eta_1 + \eta_1^* + R}},$$

$$e^R = \frac{\mu(|\alpha|^2 + |\beta|^2)}{(k_1 + k_1^*)^2}, \tag{3}$$

where

$$\eta_j = k_j(t + ik_j x) + \eta_j^{(0)}. \tag{4}$$

The arbitrary complex parameters in Eqs. (3) are k_1 and any two of the set $\alpha, \beta, \eta_1^{(0)}$. This solution can also be written in the more conventional form

$$q_i = \frac{k_{1R} A_i e^{i\eta_{1I}}}{\cosh(\eta_{1R} + \phi)}, \tag{5}$$

where

$$\phi = \frac{1}{2} R, \quad A_1 = \frac{\alpha}{\Delta}, \quad A_2 = \frac{\beta}{\Delta}, \quad \Delta = \sqrt{\mu(|\alpha|^2 + |\beta|^2)},$$

and we have introduced the subscripts R and I for the real and imaginary parts of the quantity in question. (A positive root has been used to define $e^{R/2}$ and hence ϕ is real.) From this form it is easy to identify the amplitude A and the phase

ϕ. We also note that k_{1R} and k_{1I} determine the amplitude and velocity of solitons, and in Manakov's notation [8] A_i corresponds to the unit polarization vector of the soliton.

The two-soliton solution can be obtained by substituting into Eqs. (2) the expansion $g = \chi g_1 + \chi^3 g_3$, $h = \chi h_1 + \chi^3 h_3$, and $f = 1 + \chi^2 f_2 + \chi^4 f_4$, where χ is the formal expansion parameter. The main problem is to choose the proper ansatz for g_1, h_1. In [10] a bright two-soliton solution was derived in this way assuming the input expression $g_1 = e^{\eta_1} + e^{\eta_2}$, $h_1 = e^{\varepsilon}(e^{\eta_1} + e^{\eta_2})$. The resulting two-soliton solution shows the standard elastic collision with a phase shift.

However, for any integrable equation it must be possible to combine *any* pair of one-soliton solutions into a two-soliton solution [16], and therefore it should also be possible to start with g_1 and h_1 given by

$$g_1 = \alpha_1 e^{\eta_1} + \alpha_2 e^{\eta_2}, \quad h_1 = \beta_1 e^{\eta_1} + \beta_2 e^{\eta_2}, \tag{6}$$

and this way generate a more general bright two-soliton solution with *six* arbitrary complex parameters k_1, k_2, α_1, α_2, β_1, and β_2. [Note that the parameters $\eta_j^{(0)}$ in Eqs. (4) have been absorbed into α_j and β_j.] This is indeed possible. By following Eqs. (10)–(28) in [10], we obtain the most general expressions for the two optical modes q_1 and q_2 as

$$q_1 = \frac{\alpha_1 e^{\eta_1} + \alpha_2 e^{\eta_2} + e^{\eta_1 + \eta_1^* + \eta_2 + \delta_1} + e^{\eta_1 + \eta_2 + \eta_2^* + \delta_2}}{1 + e^{\eta_1 + \eta_1^* + R_1} + e^{\eta_1 + \eta_2^* + \delta_0} + e^{\eta_1^* + \eta_2 + \delta_0^*} + e^{\eta_2 + \eta_2^* + R_2} + e^{\eta_1 + \eta_1^* + \eta_2 + \eta_2^* + R_3}},$$

$$q_2 = \frac{\beta_1 e^{\eta_1} + \beta_2 e^{\eta_2} + e^{\eta_1 + \eta_1^* + \eta_2 + \delta_1'} + e^{\eta_1 + \eta_2 + \eta_2^* + \delta_2'}}{1 + e^{\eta_1 + \eta_1^* + R_1} + e^{\eta_1 + \eta_2^* + \delta_0} + e^{\eta_1^* + \eta_2 + \delta_0^*} + e^{\eta_2 + \eta_2^* + R_2} + e^{\eta_1 + \eta_1^* + \eta_2 + \eta_2^* + R_3}}, \tag{7}$$

where (note that R_i are real)

$$e^{\delta_0} = \frac{\kappa_{12}}{k_1 + k_2^*}, \quad e^{R_1} = \frac{\kappa_{11}}{k_1 + k_1^*}, \quad e^{R_2} = \frac{\kappa_{22}}{k_2 + k_2^*}, \tag{8}$$

$$e^{\delta_1} = \frac{k_1 - k_2}{(k_1 + k_1^*)(k_1^* + k_2)}(\alpha_1 \kappa_{21} - \alpha_2 \kappa_{11}), \tag{9}$$

$$e^{\delta_2} = \frac{k_2 - k_1}{(k_2 + k_2^*)(k_1 + k_2^*)}(\alpha_2 \kappa_{12} - \alpha_1 \kappa_{22}), \tag{10}$$

$$e^{\delta_1'} = \frac{k_1 - k_2}{(k_1 + k_1^*)(k_1^* + k_2)}(\beta_1 \kappa_{21} - \beta_2 \kappa_{11}), \tag{11}$$

$$e^{\delta_2'} = \frac{k_2 - k_1}{(k_2 + k_2^*)(k_1 + k_2^*)}(\beta_2 \kappa_{12} - \beta_1 \kappa_{22}), \tag{12}$$

$$e^{R_3} = \frac{|k_1 - k_2|^2}{(k_1 + k_1^*)(k_2 + k_2^*)|k_1 + k_2^*|^2}(\kappa_{11}\kappa_{22} - \kappa_{12}\kappa_{21}), \tag{13}$$

and

$$\kappa_{ij} = \frac{\mu(\alpha_i \alpha_j^* + \beta_i \beta_j^*)}{k_i + k_j^*}. \tag{14}$$

Does the introduction of the additional parameters make any qualitative change in the behavior of soliton? The answer is yes and we find the novel result that the above general solution (7) corresponds to an *inelastic* collision of two bright solitons, as long as $\alpha_1 : \alpha_2 \neq \beta_1 : \beta_2$. In order to see this, we analyze the asymptotic form of the solution (7).

One important advantage of Hirota's method is that the solution allows easy analysis of the asymptotic behavior by taking a given $\eta_{iR} \to \pm\infty$ in Eqs. (7) and comparing the result with the one-soliton solution (3). The interpretation of the result in terms of the actual motion of the solitons depends on the signs of k_{iR} and k_{iI}. In general $\eta_{iR} = k_{iR}(t - 2k_{iI}x)$ so that soliton j is located in the vicinity of the line $t = 2k_{jI}x$. Let us change to the frame comoving with soliton j (coordinatized by ξ) by putting $x = (t - \xi)/(2k_{jI})$. Then $\eta_{jR} = k_{jR}\xi$, while for the other soliton m we get $\eta_{mR} = k_{mR}(1 - k_{mI}/k_{jI})t + k_{mR}(k_{mI}/k_{jI})\xi$.

Thus, if we have, for example, $k_{iR} > 0$ and $k_{1I}k_{2I} < 0$, which corresponds to a head-on collision, we find that $\eta_{iR} \to \pm\infty$ corresponds to $t \to \pm\infty$. (Correspondence between the signs of $x \to \pm\infty$ and $\eta_{iR} \to \pm\infty$ is obtained in a similar way and leads to a slightly different dependence on the signs of the k's.)

In each limit $\eta_{iR} \to \pm\infty$ the resulting q_i can be written as in Eq. (5) with different amplitudes A and phases ϕ. Let us denote by $A_i^{k\pm}$ the amplitude of the component q_i of the soliton k as the other soliton goes to $\pm\infty$, and similarly for the phase ϕ. Furthermore, let us define the phase shift $\Phi^k = \phi^{k+} - \phi^{k-}$, and a "transition matrix" T_i^k by $A_i^{k+} = A_i^{k-}T_i^k$. (Since the magnitudes of A and T are the most interesting quantities, we will not study their phases here.) We find the following results.

Soliton 1:

$$|A_1^{1-}| = |\alpha_1|/\Delta_1, \quad |A_2^{1-}| = |\beta_1|/\Delta_1, \quad (15)$$

$$|T_1^1| = \frac{|1 - \lambda_2\alpha_2/\alpha_1|}{\sqrt{|1 - \lambda_1\lambda_2|}}, \quad |T_2^1| = \frac{|1 - \lambda_2\beta_2/\beta_1|}{\sqrt{|1 - \lambda_1\lambda_2|}}, \quad (16)$$

$$\phi^{1-} = \frac{1}{2} R_1, \quad \Phi^1 = \frac{1}{2} (R_3 - R_1 - R_2). \quad (17)$$

Soliton 2:

$$|A_1^{2-}| = |\alpha_2|/\Delta_2, \quad |A_2^{2-}| = |\beta_2|/\Delta_2, \quad (18)$$

$$|T_1^2| = \frac{|1 - \lambda_1\alpha_1/\alpha_2|}{\sqrt{|1 - \lambda_1\lambda_2|}} \quad |T_2^2| = \frac{|1 - \lambda_1\beta_1/\beta_2|}{\sqrt{|1 - \lambda_1\lambda_2|}}, \quad (19)$$

$$\phi^{2-} = \frac{1}{2} R_2, \quad \Phi^2 = \frac{1}{2} (R_3 - R_1 - R_2)(=\Phi^1), \quad (20)$$

where

$$\Delta_i = \sqrt{\mu(|\alpha_i|^2 + |\beta_i|^2)}, \quad \lambda_1 = \kappa_{21}/\kappa_{11}, \quad \lambda_2 = \kappa_{12}/\kappa_{22}. \quad (21)$$

The expressions above are scale invariant in the way that, for example, allows us to take $\beta_i = 1$. If we also have $\alpha_1 = \alpha_2$, then one can easily verify that $|T_i^j| = 1$, which implies perfect elastic scattering. We also note that the noninteracting stationary pulses discussed in [17] follow with the choice $\alpha_2 = \beta_1 = 0, k_{1I} = k_{2I} = 0$.

One can verify that $|A_1^{i-}|^2 + |A_2^{i-}|^2 = |A_1^{i+}|^2 + |A_2^{i+}|^2 = 1/\mu$ so that the *total* intensity of each soliton is conserved. However, the *distribution* of this intensity among the two component fields can change during collision. It turns out that an inelastic effect can be obtained just by changing the relative *phases* of the parameters α_i, β_i.

In Fig. 1 we have a picture of a head-on collision with $k_1 = 1 + i$, $k_2 = 2 - i$, $\beta_i = 1$, $\alpha_1 = 1$, and $\alpha_2 = (39 + i80)/89$. We still have $|\alpha_2| = 1$, but its nontrivial phase ($\alpha_2 = e^{i\theta}$, $\theta \approx 64°$) is enough to cause quite dramatic non-elasticity. The initial time profiles at both ends of the x axis are evenly split between the two components, but the profile observed later at the large positive x end is almost completely in component 2. If we had chosen $\alpha_2 = 1$, the scatter-
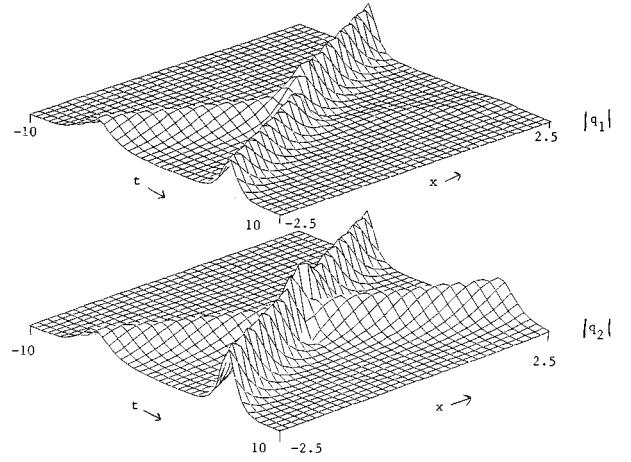

FIG. 1. Intensity profiles $|q_1|$ (top) and $|q_2|$ (bottom) of the head-on collision solution (7) with the parameter values $k_1 = 1 + i$, $k_2 = 2 - i$, $\beta_1 = \beta_2 = \alpha_1 = 1$, and $\alpha_2 = (39 + i80)/89$.

ing would have been elastic. Thus, by changing the relative phase of just one component of one soliton, we are able to change the final state quite dramatically.

Finally, we wish to connect explicitly our results on inelastic collision with that of the work of Manakov in [8]. In formulating the results that he obtained by an asymptotic analysis of the inverse scattering problem associated with the system (1), Manakov used a (complex unit) polarization vector obtained from the amplitudes of the solitons [Eqs. (19) and (20) in [8]]. Correspondingly, in our case the initial polarization vectors are

$$\hat{c}_i = (\alpha_i, \beta_i)^T / \sqrt{|\alpha_i|^2 + |\beta_i|^2}, \quad (22)$$

while the final vectors can be expressed as

$$\hat{c}_1' = \frac{1}{\chi} \frac{k_1 - k_2}{k_1 + k_2^*} \left(\hat{c}_1 - \frac{k_2 + k_2^*}{k_1 + k_2^*} (\hat{c}_1 \cdot \hat{c}_2^*) \hat{c}_2 \right), \quad (23)$$

$$\hat{c}_2' = \frac{1}{\chi} \frac{k_2 - k_1}{k_2 + k_1^*} \left(\hat{c}_2 - \frac{k_1 + k_1^*}{k_2 + k_1^*} (\hat{c}_2 \cdot \hat{c}_1^*) \hat{c}_1 \right), \quad (24)$$

where

$$\chi = \frac{|k_1 - k_2|}{|k_1 + k_2^*|} \left\{ 1 - \frac{(k_1 + k_1^*)(k_2 + k_2^*)}{|k_1 + k_2^*|^2} \left| \hat{c}_1 \cdot \hat{c}_2^* \right|^2 \right\}^{1/2}. \quad (25)$$

[The connection between Manakov's ζ and our k is $\zeta_1 = (-ik_2)^*$, $\zeta_2 = -ik_2$.]

Manakov pointed out that during soliton collision their velocities and amplitudes (intensities) do not change but the associated unit polarization vectors do change provided they are neither parallel nor orthogonal. Our observation, illustrated in Fig. 1, is that even if we keep the initial *amplitudes* unchanged and just change their relative *phases* we can change the amplitude distribution in the final polarization vector. The parametric choice associated with Fig. 1 is $\hat{c}_1 = (1/\sqrt{2}, 1/\sqrt{2})^T$, $\hat{c}_2 = (-(39 + 80i)/(89\sqrt{2}), 1/\sqrt{2})^T$, $\zeta_1 = 1 + i$, $\zeta_2 = -1 - 2i$. Each component here has magnitude $1/\sqrt{2}$. After collision the unit polarization vectors take the

values $\hat{c}_1' = (-0.0190 + 0.0593i, -0.734 + 0.677i)$ and $\hat{c}_2' = (0.422 + 0.272i, 0.462 - 0.731i)$, with magnitude distributions $(0.0623, 0.998)$ and $(0.502, 0.865)$, respectively. One can also note that the magnitude of the first component of $\hat{c}_1'$ can be even made exactly zero if we can also change the parameters k_i suitably. However, in practical applications it is probably only the relative phase that can be easily modified.

To conclude, the general two-soliton solution (7) of the $(1+1)$-dimensional system (1) exhibits a novel type of inelastic collision, not seen in any other $(1+1)$-dimensional evolution equation. Naturally such a property will have important ramifications in optical fiber communication such as providing intensity pump sources, soliton switching, and so on, which remain to be explored.

It will also be of interest to investigate the ramifications of the above type of inelastic collision in nonintegrable cases, for example, when the nonlinear cross coupling coefficients are different from one.

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[1] G. P. Agrawal, *Nonlinear Fiber Optics* (Academic Press, New York, 1995); F. Abdullaev, S. Darmanyan, and P. Khabibullaev, *Optical Solitons* (Springer-Verlag, Berlin, 1993).

[2] Y. Kodama, A. Maruta, and A. Hasegawa, Quantum Opt. **6**, 463 (1994).

[3] K. J. Blow and N. J. Doran, Electron. Lett. **19**, 429 (1983); T. Georges and F. Favre, Opt. Lett. **16**, 1656 (1991).

[4] V. E. Zakharov and A. B. Shabat, Zh. Eksp. Teor. Fiz. **61**, 118 (1971) [Sov. Phys. JETP **34**, 62 (1972)]; T. Okamawari and A. Hasegawa, Phys. Rev. A **51**, 3203 (1995); I. M. Uzunov, V. S. Gerdjikov, M. Gölles, and F. Lederer, Opt. Commun. **125**, 237 (1996).

[5] T. Ueda and W. L. Kath, Phys. Rev. A **42**, 563 (1990); Y. S. Kivshar and S. K. Turitsyn, Opt. Lett. **18**, 337 (1993).

[6] C. R. Menyuk, Opt. Lett. **12**, 614 (1987).

[7] B. Crosignani and P. Di Porto, Opt. Lett. **6**, 329 (1981).

[8] S. V. Manakov, Zh. Eksp. Teor. Fiz. **65**, 505 (1973) [Sov. Phys. JETP **38**, 248 (1974)].

[9] D. J. Kaup and B. A. Malomed, Phys. Rev. A **48**, 599 (1993).

[10] R. Radhakrishnan and M. Lakshmanan, J. Phys. A **28**, 2683 (1995).

[11] M. V. Tratnik and J. E. Sipe, Phys. Rev. A **38**, 2011 (1988).

[12] P. K. A. Wai, C. R. Menyuk, and H. H. Chen, Opt. Lett. **16**, 1231 (1991); C. De Angelis, S. Wabnitz, and M. Haelterman, Electron. Lett. **29**, 1568 (1993); C. De Angelis and S. Wabnitz, Opt. Commun. **125**, 186 (1996); P. K. A. Wai and C. R. Menyuk, J. Lightwave Technol. **14**, 148 (1996); L. F. Mollenauer, J. P. Gordon, and F. Heismann, Opt. Lett. **20**, 2060 (1995).

[13] P. A. Bélanger and C. Paré, Phys. Rev. A **41**, 5254 (1990).

[14] A. V. Buryak, Y. S. Kivshar, and D. F. Parker, Phys. Lett. A **215**, 57 (1996).

[15] R. Hirota, J. Math. Phys. (N.Y.) **14**, 805 (1973).

[16] J. Hietarinta, in *Partially Integrable Evolution Equations in Physics*, edited by R. Conte and N. Boccara, Vol. 310 of *NATO Advanced Study Institute, Series C* (Kluwer, Dordrecht, 1990), p. 459.

[17] M. Karlsson, D. J. Kaup, and B. A. Malomed, Phys. Rev. E **54**, 5802 (1996).

[18] M. Boiti, J. J.-P. Leon, L. Martina, and F. Pempinelli, Phys. Lett. A **132**, 432 (1988).

[19] A. S. Fokas and P. M. Santini, Physica D **44**, 99 (1990).

[20] J. Hietarinta and R. Hirota, Phys. Lett. A **145**, 237 (1990); J. Hietarinta, in *Nonlinear Evolution Equations and Dynamical Systems, NEEDS '91*, edited by M. Boiti *et al.* (World Scientific, Singapore, 1992), p. 23.